# Bosonization of Chiral Fermions in One Spatial Dimension

**GIRISH SETLUR**

**Department of Physics**

**IIT Guwahati**

# Fermion creation and annihilation

In many body physics, we are usually interested in the behavior of a system containing many particles when subject to external disturbances.

One very revealing way of probing the nature of the many body state of a given system is to remove a particle (usually a fermion) from the system.

This disturbs the system, as this system has to now antisymmetrise its many body wavefunction with one less fermion.

The state of the system with one less fermion ceases to be stationary and it evolves with time.

We then replace the fermion we had removed. The system now is going to be in a very different state that the original one we started off with since it was evolving in a non-stationary way when there was one less fermion.

Comparing this new state and the starting state (both of which have the same number of particles) reveals a great deal about the nature of the quasiparticles in the system.

Quasiparticles are fermions which appear to be the particles that the system is made of when probed rather than the original microscopic entities the system is actually made of.

The overlap between the new state and the starting state containing the same number of particles is known as the single-particle Green function  - quantity of central interest to the subject.

Creating and annihilating fermions are accomplished by operators denoted by

$$c_p^\dagger \quad and \quad c_p$$

These obey the following algebraic properties known as (fermion) commutation rules.

$$\{c_p\,, c_{p'}\} = 0 \qquad \{c_p^\dagger\,, c_{p'}^\dagger\} = 0$$

$$\{c_p\,, c_{p'}^\dagger\} = \delta_{p,p'}$$

where

$$\{A\,, B\} \equiv A\,B + B\,A$$

# Chiral Fermions

Chiral fermions are fermions that come with an additional discrete index similar to spin projection.

They are called right movers and left movers – similar to up spin and down spin.

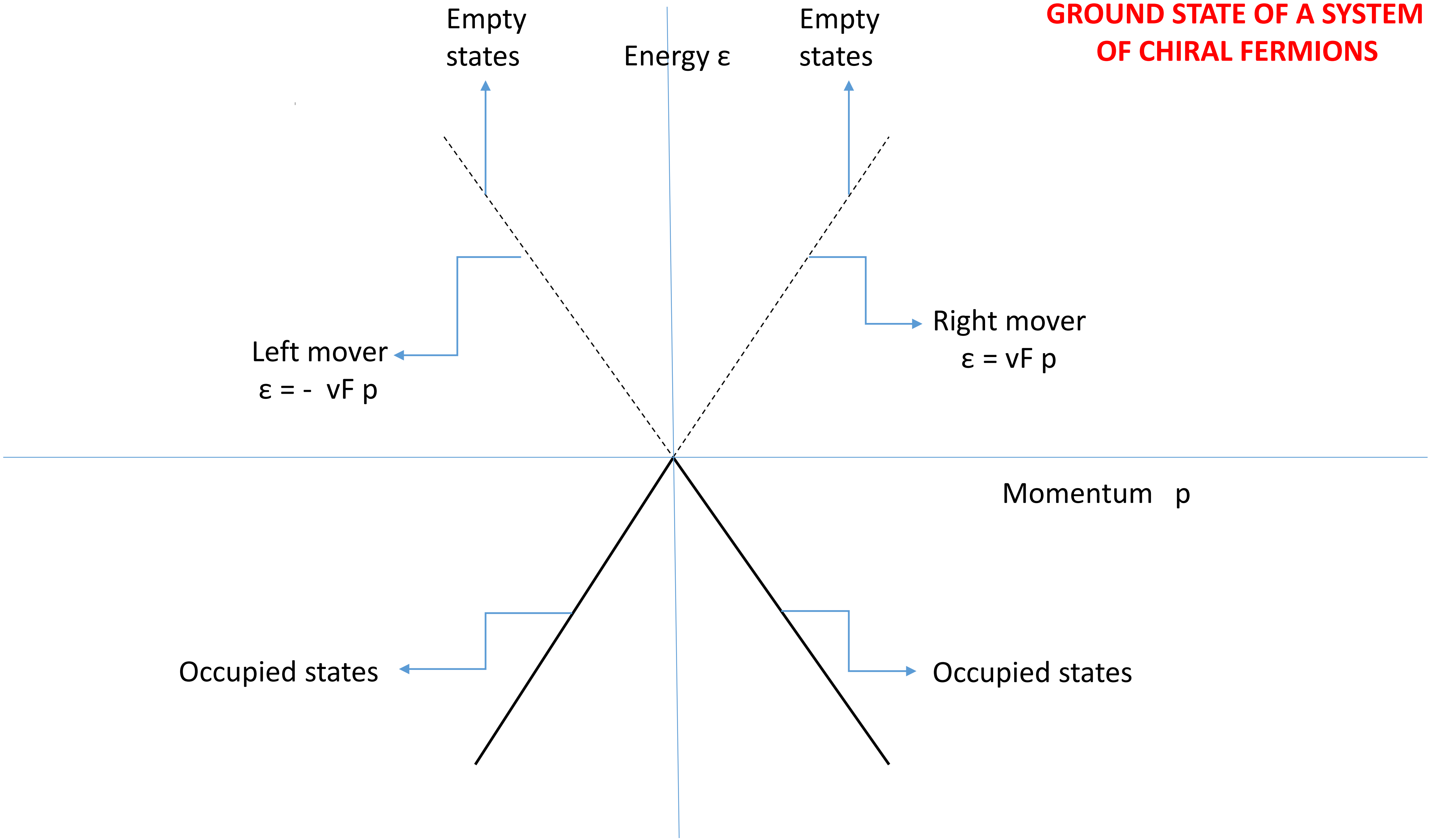
The right and left movers are postulated to move with a constant speed independent of their momentum.

This means their (kinetic) energies are proportional to their respective momenta.

The right movers have energy $+\mathrm{v}_F\, p$ and the left movers have energy $-\mathrm{v}_F\, p$
where p is the momentum of the right and left moving fermion and $\mathrm{v}_F$ is the magnitude of the speed known as Fermi velocity.

This speed is independent of the momentum and is analogous to the speed of light in relativity. The fermions have energy proportional to the momentum is anaglous to massless particles in relativistic theories.

## GROUND STATE OF A SYSTEM OF CHIRAL FERMIONS



# The Hamiltonian of free chiral fermions

$Right\ movers\ :\ c_{p,R}^{\dagger} \quad and \quad c_{p,R}$

$Left\ movers\ \ :\ c_{p,L}^{\dagger} \quad and \quad c_{p,L}$

$$H = \sum_{p} \mathrm{v}_F\ p\ \ c_{p,R}^{\dagger} c_{p,R} - \sum_{p} \mathrm{v}_F\ p\ \ c_{p,L}^{\dagger} c_{p,L}$$

# Fermion commutation rules:

$$\{c_{p,R}\,, c_{p',R}\} = \{c_{p,R}\,, c_{p',L}\} = \{c_{p,L}\,, c_{p',R}\} = \{c_{p,L}\,, c_{p',L}\} = 0$$

$$\{c^{\dagger}_{p,R}\,, c^{\dagger}_{p',R}\} = \{c^{\dagger}_{p,R}\,, c^{\dagger}_{p',L}\} = \{c^{\dagger}_{p,L}\,, c^{\dagger}_{p',R}\} = \{c^{\dagger}_{p,L}\,, c^{\dagger}_{p',L}\} = 0$$

$$\{c_{p,R}\,, c^{\dagger}_{p',R}\} = \{c_{p,L}\,, c^{\dagger}_{p',L}\} = \delta_{p,\,p'}$$

$$\{c_{p,R}\,, c^{\dagger}_{p',L}\} = \{c_{p,L}\,, c^{\dagger}_{p',R}\} = 0$$

# Real Space Forms

*We assume anti periodic boundary conditions:* $\psi_\nu(x+L) = -\psi_\nu(x)$

*where* $\nu = R, L$

$$\psi_\nu(x) = \frac{1}{\sqrt{L}} \sum_p e^{-i\,p\,x}\; c_{p,\nu}$$

$$p = \pm\frac{\pi}{L}\,, \pm\frac{3\pi}{L}, \pm\frac{5\pi}{L}\,, \ldots$$

# Real space commutation rules

$$\{\psi_\nu(x)\,,\psi_{\nu'}(x')\} = 0$$

$$\left\{\psi_\nu(x)\,,\psi^\dagger_{\nu'}(x')\right\} = \delta_{\nu,\nu'}\;\delta(x-x')$$

where $\delta(x-x')$ is the antiperiodic Dirac Delta function.

$$\delta(x-x') = \frac{1}{L}\sum_p e^{-i\,p\,(x-x')}$$

# Density of chiral fermions: Normal ordering

Naively we think of density of quantum particles being related to the field as follows:

$$\rho_\nu(x) = \psi_\nu^\dagger(x)\psi_\nu(x)$$

The problem with definition is that the average of this operator diverges. This is understandable since the number of right movers are infinite as are the number of left movers even though the length of the system is fixed.

To remedy this we definite instead the density fluctuation which is the deviation from the average density. But instead of sticking with the infinite average density we employ a device known as "point splitting" which means writing:

$$\rho_\nu(x) = Lim_{a \to 0}\left( \psi_\nu^\dagger(x+a)\psi_\nu(x) - < \psi_\nu^\dagger(x+a)\psi_\nu(x) >_0 \right)$$

Note that while $< \psi_\nu^\dagger(x)\psi_\nu(x) >_0$ is infinite, $< \psi_\nu^\dagger(x+a)\psi_\nu(x) >_0$ is finite and the above expression is mathematically well defined. It is possible to define the Fourier components of the density also. This is easier to write down mathematically as issues such as point splitting are important in real space.

We write $(\nu$ = R,L),

$$\rho_{q>0,R} = \sum_k c_{k,R}^\dagger \, \mathrm{c}_{k+q,R}$$
$$\rho_{q<0,L} = \sum_k c_{k,L}^\dagger \, \mathrm{c}_{k+q,L}$$

$$k = \pm\frac{\pi}{L}, \pm\frac{3\pi}{L}, \ldots..$$

$$q = \pm\frac{2\pi}{L}, \pm\frac{4\pi}{L}, \ldots.$$

# Chiral anomaly

Normally, we expect density at different point but at equal times to commute. This is because for ordinary (not chiral) fermions we are accustomed to thinking of density in momentum space as,

$$\rho_q = \sum_n e^{-i\,q\,x_n}$$

where $x_n$ is the position of the n-th fermion. Since the positions at equal times commute amongst themselves we expect the following result,

$$[\rho_q\,,\rho_{q'}] = 0$$

But this is not always the case for chiral fermions. To see why, let us try to compute,

$$Q_n \equiv [\rho_{q,\nu} , \rho_{q',\nu'}]$$

Instead of actually computing this, first let us show that this quantity is proportional to the identity operator. The way to do this is to show that it commutes with all other operators in the problem, specifically,

$$c_{p,R} , c_{p,L} , \; c^{\dagger}_{p,R} , c^{\dagger}_{p,L}$$

$$[c_{p,R} , Q_n ] = \left[[c_{p,R} , \rho_{q,\nu} ], \rho_{q',\nu'}\right] + [\rho_{q,\nu} , [c_{p,R} , \rho_{q',\nu'}]]$$

$$[c_{p,R} , \rho_{q,\nu} ] = \theta(\nu\, q)\delta_{\nu,R} \; \mathrm{c}_{p+q,R}$$
$$[c_{p,R} , \rho_{q',\nu'} ] = \theta(\nu' q')\delta_{\nu',R} \; \mathrm{c}_{p+q',R}$$

$$[c_{p,R}, Q_n] = \delta_{\nu,R}\theta(q)[c_{p+q,R}, \rho_{q',\nu'}] + [\rho_{q,\nu}, c_{p+q',R}]\delta_{\nu',R}\theta(q')$$

or,

$$[c_{p,R}, Q_n] = \delta_{\nu,R}\theta(q)\theta(q')\,\delta_{\nu',R}\; c_{p+q+q',R} - \delta_{\nu,R}\delta_{\nu',R}\theta(q)\theta(q')\, c_{p+q'+q,R} = 0$$

Similarly,

$$[c^{\dagger}_{p,R}, Q_n] = [[c^{\dagger}_{p,R}, \rho_{q,\nu}], \rho_{q',\nu'}] + [\rho_{q,\nu}, [c^{\dagger}_{p,R}, \rho_{q',\nu'}]]$$

$$[c^{\dagger}_{p,R}, \rho_{q,\nu}] = -\delta_{\nu,R}\theta(q)c^{\dagger}_{p-q,R}$$

$$[c^{\dagger}_{p,R}, \rho_{q',\nu'}] = -\delta_{\nu',R}\theta(q')c^{\dagger}_{p-q',R}$$

$$[c^\dagger_{p,R}, Q_n] = -\delta_{\nu,R}\theta(q)[c^\dagger_{p-q,R}, \rho_{q',\nu'}] - \delta_{\nu',R}\theta(q')\left[\rho_{q,\nu}, c^\dagger_{p-q',R}\right]$$

but

$$\left[c^\dagger_{p-q',R}, \rho_{q,\nu}\right] = -\delta_{\nu,R}\theta(q)c^\dagger_{p-q'-q,R}$$

$$\left[c^\dagger_{p-q,R}, \rho_{q',\nu'}\right] = -\delta_{\nu',R}\theta(q')c^\dagger_{p-q'-q,R}$$

Hence,

$$[c^\dagger_{p,R}, Q_n] = \delta_{\nu',R}\delta_{\nu,R}\theta(q)\theta(q')c^\dagger_{p-q'-q,R} - \delta_{\nu',R}\delta_{\nu,R}\theta(q)\theta(q')c^\dagger_{p-q'-q,R} = 0$$

This means we have proved $[c^\dagger_{p,R}, Q_n\,] = 0\;\; and\; [c_{p,R}\,, Q_n\,] = 0.$

Similarly we may prove $[c^\dagger_{p,L}, Q_n\,] = 0\;\; and\; [c_{p,L}\,, Q_n\,] = 0\; (homework)$

This means $Q_n$ is independent of the fermions and is proportional to the identity operator. The proportionality factor may be thought of as the average with respect to the ground state of the system.

$$Q_n = < \mathrm{Q}_n\; > \mathbf{I}$$

If |G> is the ground state of the system described by two chiral fermions with all negative energies filled and positive energies empty, by construction,

$$\rho_{q>0,R}|G> = \sum_k c^\dagger_{k,R}\; \mathrm{c}_{k+q,R}\,|G> = 0$$
$$\rho_{q<0,L}|G> = \sum_k c^\dagger_{k,L}\; \mathrm{c}_{k+q,L}\,|G> = 0$$

This means

$$< Q_n > \equiv < G\left|\left[\rho_{q,\nu}, \rho_{q',\nu'}\right]\right|G > = 0$$

Thus, $Q_n \equiv \left[\rho_{q,\nu}, \rho_{q',\nu'}\right] = < \mathrm{Q}_n > \mathbf{I} = 0$

Next we look at the commutator: $Q_a \equiv \left[\rho_{q,\nu}, \rho^{\dagger}_{q',\nu'}\right]$

We may show that this also commutes with all the fermions so that it is also proportional to the identity. But crucially, the proportionality factor which is the expectation value of the above operator with respect to the ground state is (sometimes) nonzero

**– this is known as the chiral anomaly.**

Examine the commutator, $\left[c_{p,R}\,,Q_a\right]=\left[\left[c_{p,R}\,,\rho_{q,\nu}\right],\rho^{\dagger}_{q',\nu'}\right]+\left[\rho_{q,\nu},\left[c_{p,R}\,,\rho^{\dagger}_{q',\nu'}\right]\right]$

$$\rho_{q>0,R}=\sum_k c^{\dagger}_{k,R}\;c_{k+q,R} \qquad\qquad \rho^{\dagger}_{q'>0,R}=\sum_k c^{\dagger}_{k+q',R}\,c_{k,R}$$

$$\left[c_{p,R}\,,Q_a\right]=\theta(q)\,\delta_{\nu,R}\left[c_{p+q,R},\rho^{\dagger}_{q',\nu'}\right]+\theta(q')\,\delta_{\nu',R}\left[\rho_{q,\nu},c_{p-q',R}\right]$$

$$\left[c_{p+q,R},\rho^{\dagger}_{q',\nu'}\right]=\theta(q')\,\delta_{\nu',R}\,c_{p+q-q',R}; \qquad \left[\rho_{q,\nu},c_{p-q',R}\right]=-\theta(q)\,\delta_{\nu,R}\,c_{p+q-q',R}$$

$$\left[c_{p,R}\,,Q_a\right]=\theta(q)\,\delta_{\nu,R}\theta(q')\,\delta_{\nu',R}\,c_{p+q-q',R}-\;\theta(q')\,\delta_{\nu',R}\theta(q)\,\delta_{\nu,R}\,c_{p+q-q',R}$$

or

$$\left[c_{p,R}\,,Q_a\right]=\theta(q)\,\delta_{\nu,R}\theta(q')\,\delta_{\nu',R}\,c_{p+q-q',R}-\;\theta(q')\,\delta_{\nu',R}\theta(q)\,\delta_{\nu,R}\,c_{p+q-q',R}=0$$

Thus $\left[\mathrm{c}_{p,R}\,,Q_a\right]=0$

Similarly we may conclude, $\left[\mathrm{c}^{\dagger}_{p,R}\,,Q_a\right]=0$

and $\left[\mathrm{c}_{p,L}\,,Q_a\right]=0$

and $\left[\mathrm{c}^{\dagger}_{p,L}\,,Q_a\right]=0$

This means

$$Q_a \;=\; <Q_a>\mathbf{I}$$

But,

$$< Q_a > = < G \left| \left[ \rho_{q,\nu} , \rho^{\dagger}_{q',\nu'} \right] \right| G > = < G \left| \rho_{q,\nu} \, \rho^{\dagger}_{q',\nu'} \right| G >$$

or,

$$< Q_a > \; = \; < G \left| \rho_{q,\nu} \, \rho^{\dagger}_{q',\nu'} \right| G > \; =$$

$$\delta_{q,q'} \delta_{\nu,\nu'} \, \theta(q) \, \delta_{\nu,R} \sum_k < G | c^{\dagger}_{k,R} \; c_{k+q,R} \; c^{\dagger}_{k+q,R} \; c_{k,R} | G >$$

$$+ \delta_{q,q'} \delta_{\nu,\nu'} \, \theta(-q) \, \delta_{\nu,L} \sum_k < G | c^{\dagger}_{k,L} \; c_{k+q,L} \; c^{\dagger}_{k+q,L} \; c_{k,L} | G >$$

$$= \delta_{q,q'} \delta_{\nu,\nu'} \left( \theta(q) \, \delta_{\nu,R} \sum_{-q<k<0} 1 + \theta(-q) \, \delta_{\nu,L} \sum_{-q>k>0} 1 \right)$$

$$= \delta_{q,q'} \delta_{\nu,\nu'} \frac{L}{2\pi} \left( \theta(q) \, \delta_{\nu,R} \, q - \theta(-q) \, \delta_{\nu,L} \, q \right)$$

$$< Q_a > \; = \delta_{q,q'}\delta_{\nu,\nu'}\frac{\nu\, q\, L}{2\pi}\theta(\nu\, q)$$

This means,

$$\left[\rho_{q,\nu}\,,\rho_{q',\nu'}\right] = 0 \quad \text{and} \quad \left[\rho_{q,\nu}\,,\rho^{\dagger}_{q',\nu'}\right] = \delta_{q,q'}\delta_{\nu,\nu'}\frac{\nu\, q\, L}{2\pi}\theta(\nu\, q)$$

Chiral Anomaly

# Recap

- We defined chiral fermions
- We defined density of right movers and left movers
- We showed that unlike ordinary fermions the density operators of chiral fermions don’t commute amongst themselves.
- We derived explicit forms of these commutators and showed they are proportional to the identity operator when they are not zero.

# Bosonization of chiral fermions:

Using the chiral anomaly idea, we may define canonical boson creation and annihilation operators. Set

$$b_{q>0,R} = \sqrt{\frac{2\pi}{qL}}\ \rho_{q,R}$$

$$b_{q<0,L} = \sqrt{\frac{2\pi}{-qL}}\ \rho_{q,L}$$

These obey canonical boson commutation rules:

$$[b_{q,R}, b_{q',R}] = [b_{q,R}, b_{q',L}] = [b_{q,L}, b_{q',R}] = [b_{q,L}, b_{q',L}] = 0$$

$$[b_{q,R}, b^{\dagger}_{q',L}] = [b_{q,L}, b^{\dagger}_{q',R}] = 0$$

$$[b_{q,R}, b^{\dagger}_{q',R}] = [b_{q,L}, b^{\dagger}_{q',L}] = 1$$

# Normal Ordering

Recall that we said that the kinetic energy of free chiral fermions was written as

$$H = H_R + H_L$$

where

$$H_R = \sum_p \mathrm{v}_F \; p \;\; c^\dagger_{p,R} c_{p,R}$$

and

$$H_L = -\sum_p \mathrm{v}_F \; p \;\; c^\dagger_{p,L} c_{p,L}$$

The problem with this definition is that the eigenvalues of $H_R$ & $H_L$ are (negative) infinities when acting on the ground state, for example. We want to define all in such a way that in the ground state, the eigenvalues are zero instead of infinite. This is accomplished through what is known as normal ordering which is just a clever way of subtracting the infinities and only considering the excitations from the ground state.

We define normal ordered kinetic energy of right movers as:

$$:H_R: = \sum_{p>0} \mathrm{v}_F\ p\ c^\dagger_{p,R} c_{p,R} + \sum_{p>0} \mathrm{v}_F\ p\ c_{-p,R} c^\dagger_{-p,R}$$

It is easy to see that the difference between this and the original $H_R$ is an (infinite) constant.

$$:H_R: - H_R = \sum_{p>0} \mathrm{v}_F\ p\ c^{\dagger}_{p,R} c_{p,R} + \sum_{p>0} \mathrm{v}_F\ p\ c_{-p,R} c^{\dagger}_{-p,R} - \sum_{p} \mathrm{v}_F\ p\ c^{\dagger}_{p,R} c_{p,R}$$

Or,

$$:H_R: - H_R = \sum_{p>0} \mathrm{v}_F\ p\ c_{-p,R} c^{\dagger}_{-p,R} - \sum_{p<0} \mathrm{v}_F\ p\ c^{\dagger}_{p,R} c_{p,R}$$

$$= \sum_{p>0} \mathrm{v}_F\ p\ \left( c_{-p,R}\ c^{\dagger}_{-p,R} + c^{\dagger}_{-p,R}\, c_{-p,R} \right) = \sum_{p>0} \mathrm{v}_F\ p = \mathbf{I}\,\infty$$

It is easy to see that the eigenvalue of $:H_R:$ on the ground state is zero. Therefore $:H_R:$ measure the kinetic energy of excited states relative to the ground state (also known as Virasoro primary state)

$$:H_R:|G>=0$$

Similarly we may definite normal ordered left moving kinetic energy.

$$:H_L:=-\sum_{p<0} \mathrm{v}_F\ p\ c_{p,L}^{\dagger}c_{p,L}-\sum_{p<0} \mathrm{v}_F\ p\ c_{-p,L}c_{-p,L}^{\dagger}$$

$$:H_L:|G>=0$$

Similarly, we define normal ordered number of right movers as:

$$:N_R: = \sum_{p>0} ( c_{p,R}^{\dagger} \, c_{p,R} - c_{-p,R} \, c_{-p,R}^{\dagger})$$

For left movers

$$:N_L: = \sum_{p<0} ( c_{p,L}^{\dagger} \, c_{p,L} - c_{-p,L} \, c_{-p,L}^{\dagger})$$

$:N_R: |G> = 0$ and $:N_L: |G> = 0$

We may also define normal ordered momentum of right movers as:

$$:P_R:\ = \sum_{p>0} p\ (\ c^{\dagger}_{p,R}\ c_{p,R}\ +\ c_{-p,R}\ c^{\dagger}_{-p,R})$$

For left movers

$$:P_L:\ = \sum_{p<0} p\ (\ c^{\dagger}_{p,L}\ c_{p,L}\ +\ c_{-p,L}\ c^{\dagger}_{-p,L})$$

$:P_R:|G>=0$ and $:P_L:|G>=0$

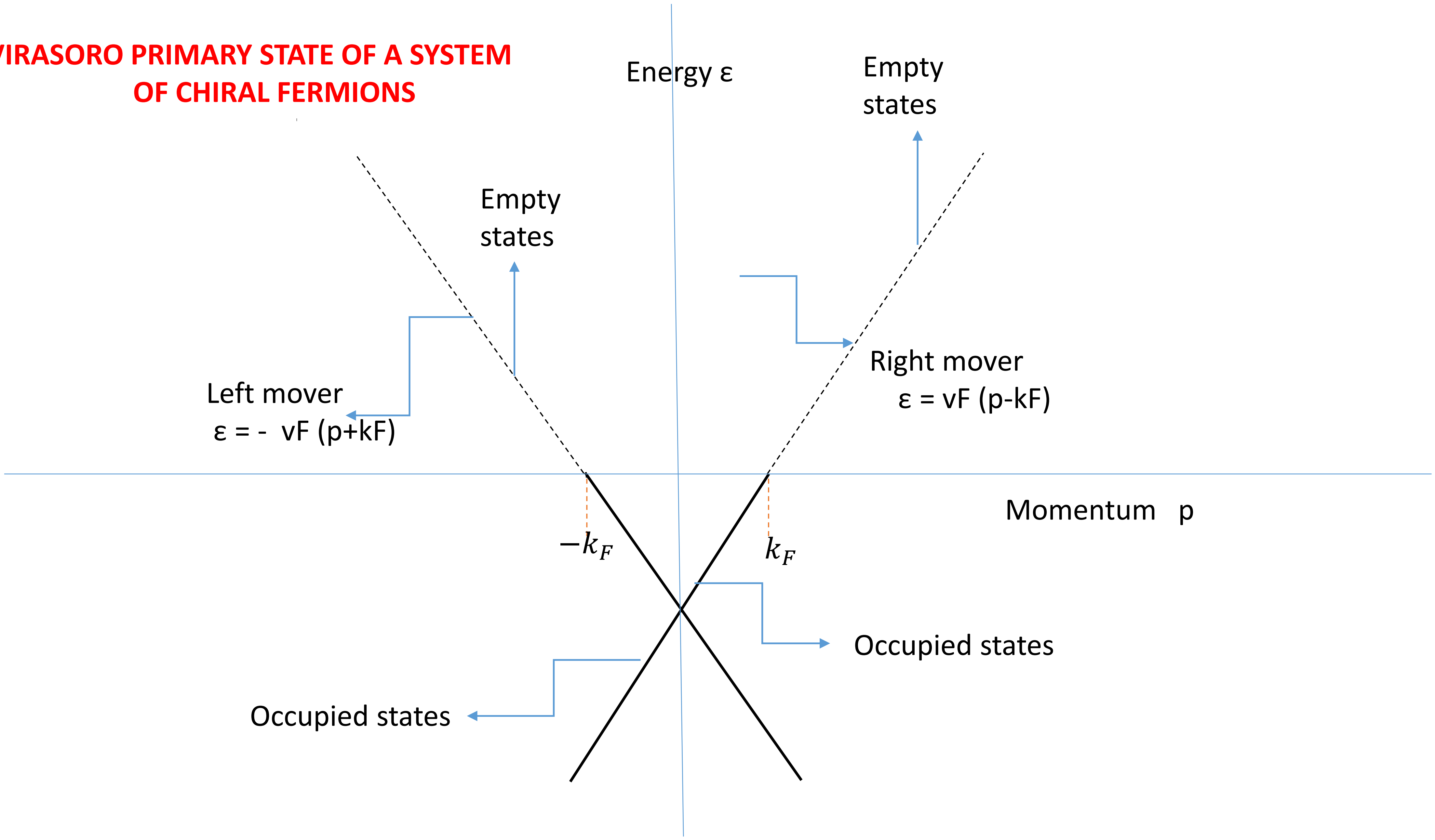
# Virasoro Primary State

For the bosonization scheme to work, it is sufficient for the ground state to be characterised by the eigenvalues of $:N_R:$, $:P_R:$ and $:N_L:$, $:P_L:$ . These eigenvalues don't have to be zero. The general "Virasoro Primary State" is the ground state characterized by the eigenvalues

$:N_R:|G> = N_R\,|G>$ and $:P_R:|G> = P_R\,|G>$

Similarly for left movers. Note that $N_R$ can be positive, negative or zero since it measures deviation from the state where all these eigenvalues vanish.

# VIRASORO PRIMARY STATE OF A SYSTEM OF CHIRAL FERMIONS



If $N_R$ is the eigenvalue of $:N_R:$ , the Virasoro Primary State has these properties ( $k_F = \frac{2\pi N_R}{L}$ )

$$c^{\dagger}_{p<k_F,R}|G> = 0 \; ; \quad c_{p>k_F,R}|G> = 0$$

$$b_{q>0,R}|\mathrm{G}> = 0$$

Using these definitions we may calculate the eigenvalue of

$$:P_R: = \sum_{p>0} p\,(c^{\dagger}_{p,R}\,c_{p,R} + c_{-p,R}\,c^{\dagger}_{-p,R})$$

Therefore,

$$:P_R:|G> \; = \sum_{p>0} p\,(\,c^{\dagger}_{p,R}\,c_{p,R}|G> \; + c_{-p,R}\,c^{\dagger}_{-p,R}|G>)$$

Now,

$$c^{\dagger}_{p,R}\,c_{p,R}\,|G> = \; \theta(\mathrm{k}_F - p)\,|G>$$

$$c_{-p,R}\,c^{\dagger}_{-p,R}|G> = \; \theta(-p - k_F)|G>$$

For example, assume $\mathrm{k}_F > 0$, in this case only the first term survives. $:P_R:|G> \; = \sum_{k_F > \, p>0} p\,|G>$

But $\sum_{k_F > p > 0} p = \frac{L}{2\pi} \int_0^{k_F} p \, dp = \frac{L}{2\pi} \frac{k_F^2}{2}$

$$= \frac{L}{2\pi} \frac{1}{2} \frac{2\pi N_R}{L} \frac{2\pi N_R}{L} = \pi \frac{N_R^2}{L}$$

This means the eigenvalue of the momentum is

$$: P_R : |G> = \frac{\pi N_R^2}{L} |G>$$

# Expressing Kinetic energy & Momentum of Chiral Fermions in Terms of Bosons

The central claim of bosonization is that all operators and properties that can be described/computed using the right (left) moving fermion operators $c^\dagger_{p,R}\,,c_{p,R}$ (or $c^\dagger_{p,L}\,,c_{p,L}$ )may equally well be described and computed using the boson operators $b^\dagger_{q>0,R}\,,b_{q>0,R}$ (or $b^\dagger_{q<0,L}\,,b_{q<0,L}$)

The goal now is to express $:H_R:$ (or $:H_L:$)
in terms of the boson operators $b^\dagger_{q>0,R}\, , b_{q>0,R}$
(or $b^\dagger_{q<0,L}\, , b_{q<0,L}$).

But first we want to prove the following intuitively obvious results:

$[b^\dagger_{q>0,R}\, , :N_R:] = 0$ & $[\, b_{q>0,R}, :N_R:] = 0$

and $[:H_R:, :N_R:] = 0\, ; [:P_R:, :N_R:] = 0$

To prove $[b^{\dagger}_{q>0,R}, :N_R:] = 0$ we first show that this commutator commutes with all the fermions. This means that it is proportional to the identity operator. Then we show that the proportionality factor is in fact zero. This somewhat roundabout method is needed for chiral fermions since the presence of anomalies may lead us to draw incorrect conclusions, if we are not careful.

Examine,

$$[c_{k,R},[b^{\dagger}_{q>0,R},:N_R:]] = [[c_{k,R},b^{\dagger}_{q>0,R}],:N_R:] + [b^{\dagger}_{q>0,R},[c_{k,R},:N_R:]]$$

$$b^{\dagger}_{q,R} = \sqrt{\frac{2\pi}{qL}}\sum_k c^{\dagger}_{k+q,R}\ \mathrm{c}_{k,R}$$

Now,

$$[c_{k,R},[b^{\dagger}_{q>0,R},:N_R:]] = \sqrt{\frac{2\pi}{qL}}\,[c_{k-q,R},:N_R:] + [b^{\dagger}_{q>0,R},c_{k,R}]$$

or,

$$[c_{k,R},[b^\dagger_{q>0,R},:N_R:]]=\sqrt{\frac{2\pi}{qL}}c_{k-q,R}-\sqrt{\frac{2\pi}{qL}}c_{k-q,R}=0$$

Similarly,

$$[c^\dagger_{k,R},[b^\dagger_{q>0,R},:N_R:]]$$
$$=[[c^\dagger_{k,R},b^\dagger_{q>0,R}],:N_R:]+[b^\dagger_{q>0,R},[c^\dagger_{k,R},:N_R:]]$$

or,

$$[c^\dagger_{k,R},[b^\dagger_{q>0,R},:N_R:]]=-[\sqrt{\frac{2\pi}{qL}}c^\dagger_{k+q,R},:N_R:]-[b^\dagger_{q>0,R},c^\dagger_{k,R}]$$

$$[c^\dagger_{k,R}, [b^\dagger_{q>0,R}, :N_R:]] = -[\sqrt{\frac{2\pi}{qL}}\, c^\dagger_{k+q,R}, :N_R:] - [b^\dagger_{q>0,R}, c^\dagger_{k,R}]$$

$$= \sqrt{\frac{2\pi}{qL}}\, c^\dagger_{k+q,R} - \sqrt{\frac{2\pi}{qL}}\, c^\dagger_{k+q,R} = 0$$

Thus $[b^\dagger_{q>0,R}, :N_R:]$ commutes with all the fermions and is therefore proportional to the identity operator. As usual the proportionality factor is its expectation value.

$$[b^\dagger_{q>0,R}, :N_R:] = < \mathrm{G}|\, [b^\dagger_{q>0,R}, :N_R:] |\mathrm{G} > \ \mathbf{I}$$

But $< \mathrm{G}| \, [b^{\dagger}_{q>0,R} \, , : N_R : ] |\mathrm{G} >$

$$= < \mathrm{G}| \, b^{\dagger}_{q>0,R} : N_R : |\mathrm{G} > - < \mathrm{G}| : N_R : b^{\dagger}_{q>0,R} |\mathrm{G} >$$

$= 0$

Because,

$: N_R : |\mathrm{G} > = \mathrm{N}_R |G >$ and $< \mathrm{G}| : N_R : \; = N_R < G|$

This means

$[b^{\dagger}_{q>0,R} \, , : N_R : ] = 0$ which also means

$[b_{q>0,R} \, , : N_R : ] = 0$

Now we have to prove that

$$[:H_R:,:N_R:] = 0 \;\; \& \;\; [:P_R:,:N_R:] = 0$$

This is left as a HOMEWORK to the reader.

First prove that $[:H_R:,:N_R:]$ & $[:P_R:,:N_R:]$ commute with all the fermions and therefore proportional to the identity. Their averages are trivially zero.

The next interesting questions are $[b^{\dagger}_{q>0,R}\,,:P_R:] = ?$ & $[\,b_{q>0,R},:P_R:] = ?$

$$[b^{\dagger}_{q>0,R}\,,:P_R:] = \;[\sqrt{\frac{2\pi}{qL}}\sum\nolimits_k\; c^{\dagger}_{k+q,R}\;\; \mathrm{c}_{k,R},:P_R:]$$

or

$$\left[\sqrt{\frac{2\pi}{qL}}\sum_k c^{\dagger}_{k+q,R}\;\; \mathrm{c}_{k,R},:P_R:\right] = \sqrt{\frac{2\pi}{qL}}\sum_k c^{\dagger}_{k+q,R}\;[\mathrm{c}_{k,R},:P_R:] + \sqrt{\frac{2\pi}{qL}}\sum_k [c^{\dagger}_{k+q,R}\,,:P_R:]\mathrm{c}_{k,R}$$

or,

$$[\mathrm{c}_{k,R},:P_R:] = k\;\mathrm{c}_{k,R}$$

$$[c^{\dagger}_{k+q,R},:P_R:] = -(k+q)\;c^{\dagger}_{k+q,R}$$

$$[b^{\dagger}_{q>0,R}\,,:P_R:]$$

$$= \sqrt{\frac{2\pi}{qL}} \sum_k c^{\dagger}_{k+q,R}\, k\, \mathrm{c}_{k,R} - \sqrt{\frac{2\pi}{qL}} \sum_k (k+q)\, c^{\dagger}_{k+q,R}\, \mathrm{c}_{k,R}$$

or,

$$[b^{\dagger}_{q>0,R}\,,:P_R:] = -q\, b^{\dagger}_{q>0,R}$$

Similarly,

$$[b_{q>0,R}\,,:P_R:] = q\, b_{q>0,R}$$

This means we should be able to write,

$$:P_R: = \sum_{q>0} q\, b^{\dagger}_{q>0,R}\, b_{q>0,R} + \frac{\pi :N:_R^2}{L}$$

The claim is that this operator is identical to

$$:P_R: = \sum_{p>0} p\,(\, c^{\dagger}_{p,R}\, c_{p,R} \;+\; c_{-p,R}\, c^{\dagger}_{-p,R})$$

Just because $:P_R:$ in the first bosonized form properly commutes with the bosons $b^{\dagger}_{q>0,R}$, $b_{q>0,R}$ and $:N_R:$ does not mean it is going to commute properly with the fermions

We have to show that

$$\left[ c_{p,R} \,, :P_R: \right] \equiv \left[ c_{p,R} \,, \sum_{q>0} q\; b^{\dagger}_{q>0,R}\; b_{q>0,R} + \frac{\pi :N:^2_R}{L} \right] = p\; c_{p,R}$$

But,

$$\left[ c_{p,R} \,, \sum_{q>0} q\; b^{\dagger}_{q>0,R}\; b_{q>0,R} + \frac{\pi :N:^2_R}{L} \right]$$

$$= \sum_{q>0} q [c_{p,R} \,, b^{\dagger}_{q>0,R}]\; b_{q>0,R} + \sum_{q>0} q\; b^{\dagger}_{q>0,R} [c_{p,R} \,, b_{q>0,R}] + \frac{\pi [c_{p,R} \,, :N:_R \;] :N:_R}{L} + \frac{\pi :N:_R\; [c_{p,R} \,, :N:_R \;]}{L}$$

But,

$$[c_{p,R} ,: P_R:] = \sum_{q>0} q \sqrt{\frac{2\pi}{qL}}\, c_{p-q,R}\ b_{q>0,R} + \sum_{q>0} q\, b^\dagger_{q>0,R} \sqrt{\frac{2\pi}{qL}}\, c_{p+q,R} + \frac{\pi\, c_{p,R} :N:_R}{L} + \frac{\pi :N:_R\ c_{p,R}}{L}$$

Or,

$$[c_{p,R} ,: P_R:] = \sum_{q>0,k} \frac{2\pi}{L} c_{p-q,R}\, c^\dagger_{k,R} c_{k+q,R} + \sum_{q>0,k} \frac{2\pi}{L} c^\dagger_{k+q,R}\, c_{k,R} c_{p+q,R}$$
$$+ \frac{\pi\, c_{p,R} \sum_{k>0}( c^\dagger_{k,R}\, c_{k,R} - c_{-k,R}\, c^\dagger_{-k,R})}{L} + \frac{\pi \sum_{k>0}( c^\dagger_{k,R}\, c_{k,R} - c_{-k,R}\, c^\dagger_{-k,R})\ c_{p,R}}{L}$$

Is this the same as $[c_{p,R} ,: P_R:] = p\, c_{p,R}$ **??**

But the commutators

$$\left[b_{q,R}\ ,:P_R:\right] = q\ \ b_{q>0,R}$$

and

$$[:N_R:,:P_R:] = 0$$

come out right  (HOME WORK) in the bosonic language as well as the fermionic language.

$$:P_R: = \sum_{q>0} q\ b^{\dagger}_{q>0,R}\ b_{q>0,R} + \frac{\pi :N:^2_R}{L}$$

$$:H_R: = \mathrm{v}_F :P_R: = \sum\nolimits_{q>0} \mathrm{v}_F q\ b^{\dagger}_{q>0,R}\ b_{q>0,R} + \mathrm{v}_F \frac{\pi :N:^2_R}{L}$$

# Interactions between Chiral Fermions

The interaction energy between right movers is

$$\frac{1}{2}\int dx \int dx' \, \mathrm{v}(x-x') \, \rho_R(x)\rho_R(x')$$

The interaction energy between left movers is

$$\frac{1}{2}\int dx \int dx' \, \mathrm{v}(x-x') \, \rho_L(x)\rho_L(x')$$

where $\mathrm{v}(x-x')$ is the two body potential. Between right and left movers it could be something else

$$\frac{1}{2}\int dx \int dx' \, \mathrm{w}(x-x') \, (\rho_R(x)\rho_L(x') + \rho_L(x)\rho_R(x'))$$

Note that we suggested that $\rho_R(x)$ be defined as,

$$\rho_R(x) = Lim_{a\to 0}\,(\,\psi_R^\dagger(x+a)\psi_R(x) - <\psi_R^\dagger(x+a)\psi_R(x)>_0\,)$$

It is more convenient to define it in the usual way viz.

$$\rho_R(x) = \,(\,\psi_R^\dagger(x)\psi_R(x) - <\psi_R^\dagger(x)\psi_R(x)>_0\,)$$

Except the right movers have a lower energy bound which in the end tends to infinity:

$$\psi_R(x) = \frac{1}{\sqrt{L}}\sum_{p>-k_{max}} e^{-i\,p\,x}\;c_{p,R}$$

and

$$\psi_R^\dagger(x) = \frac{1}{\sqrt{L}}\sum_{p'>-k_{max}} e^{i\,p'\,x}\,c_{p',R}^\dagger$$

$$\rho_{q>0,R} = \sum_k c^{\dagger}_{k,R} \; c_{k+q,R}$$

$$\psi^{\dagger}_R(x)\psi_R(x) - < \psi^{\dagger}_R\,(x)\psi_R(x) >_0$$

$$= \frac{1}{L} \sum_{k,q \neq 0, k>-k_{max}, k+q>-k_{max}} e^{-i\,q\,x} c^{\dagger}_{k,R}\; c_{k+q,R}$$

$$+ \frac{1}{L} \sum_{k>-k_{max}} (\; c^{\dagger}_{k,R} c_{k,R} - < c^{\dagger}_{k,R} \;\; c_{k,R} >_0 \;)$$

This makes sense only if

$$< \psi_R^\dagger (x)\psi_R(x) >_0 = \frac{1}{L} \sum_{k > -k_{max}} < c_{k,R}^\dagger \; c_{k,R} >_0$$

is finite. But,

$$< c_{k,R}^\dagger \; c_{k,R} >_0 = \; \theta(k_F \; - k)$$

so that $< \psi_R^\dagger (x)\psi_R(x) >_0 = \frac{1}{L} \sum_{-k_{\max<}k<k_F}$ is finite.

This means $\rho_R(x)$ be written as

$$\rho_R(x) = \frac{1}{L} \sum_{q>0, k>-k_{max}, k+q>-k_{max}} e^{-i\,q\,x}\; c_{k,R}^{\dagger}\, c_{k+q,R}$$

$$+ \frac{1}{L} \sum_{q<0, k>-k_{max},\, k+q>-k_{max}} e^{-i\,q\,x}\; c_{k,R}^{\dagger}\, c_{k+q,R}$$

$$+ \frac{1}{L} \sum_{k>-k_{max}} \left( c_{k,R}^{\dagger} c_{k,R} - < c_{k,R}^{\dagger}\; c_{k,R} >_0 \right)$$

When taking $k_{max} \rightarrow \infty$ limit we have to ensure that

$$\frac{1}{L}\sum_{k>-k_{max}}( c^{\dagger}_{k,R}c_{k,R} -< c^{\dagger}_{k,R} \ c_{k,R} >_0 )$$

remains finite. This is going to be the case if we postulate that all states in the Hilbert space of a practical problem have the property they are fully occupied $k < -k_{\infty}$ for some large but fixed $k_{\infty}$ *even when interactions are present.*

Thus we may write,

$$\rho_R(x)$$

$$= \frac{1}{L}\sum_{q>0,k} e^{-i\,q\,x} \ c^{\dagger}_{k,R}\, c_{k+q,R} + \frac{1}{L}\sum_{q<0.k} e^{-i\,q\,x} \ c^{\dagger}_{k,R}\, c_{k+q,R} + x - independent\ terms$$

$$\rho_R(x)$$

$$= \frac{1}{L} \sum_{q>0,k} e^{-i\,q\,x}\ c^{\dagger}_{k,R}\, c_{k+q,R} + \frac{1}{L} \sum_{q>0.k} e^{i\,q\,x}\ c^{\dagger}_{k+q,R}\, c_{k,R} + x - independent\ terms$$

or

$$\rho_R(x) = \frac{1}{L} \sum_{q>0} \sqrt{\frac{qL}{2\pi}} e^{-i\,q\,x}\ b_{q,R} + \frac{1}{L} \sum_{q>0} \sqrt{\frac{qL}{2\pi}}\ e^{i\,q\,x}\, b^{\dagger}_{q,R} + x - independent\ terms$$

$$\rho_L(x) = \frac{1}{L} \sum_{q<0} \sqrt{\frac{-qL}{2\pi}} e^{-i\,q\,x}\ b_{q,L} + \frac{1}{L} \sum_{q<0} \sqrt{\frac{-qL}{2\pi}}\ e^{i\,q\,x}\, b^{\dagger}_{q,L} + x - independent\ terms$$

We assert that $\int dx'\, \mathrm{v}(x-x') = 0$ so that

$$\frac{1}{2}\int dx \int dx'\, \mathrm{v}(x-x')\, \rho_R(x)\rho_R(x')$$

$$= \frac{1}{2}\int dx \int dx'\, \mathrm{v}(x-x') \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(e^{-i\,q\,x}\, b_{q,R} + e^{i\,q\,x}\, b^{\dagger}_{q,R}\right) \frac{1}{L}\sum_{q'>0}\sqrt{\frac{q'L}{2\pi}}\left(e^{-i\,q'\,x'}\, b_{q',R} + e^{i\,q'\,x'}\, b^{\dagger}_{q',R}\right)$$

Set,

$$\mathrm{v}(x-x') = \frac{1}{L}\sum_{q>0} \mathrm{v}_q\, e^{i\,q\,(x-x')}$$

or,

$$\frac{1}{2}\int dx \int dx'\, \mathrm{v}(x-x')\, \rho_R(x)\rho_R(x')$$

$$= \frac{1}{2}\int dx \int dx'\, \mathrm{v}(x-x') \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(e^{-i\,q\,x}\, b_{q,R}\right) \frac{1}{L}\sum_{q'>0}\sqrt{\frac{q'L}{2\pi}}\left(e^{i\,q'\,x'}\, b^{\dagger}_{q',R}\right)$$

$$+\frac{1}{2}\int dx \int dx'\, \mathrm{v}(x-x') \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\left(e^{i\,q\,x}\, b^{\dagger}_{q,R}\right) \frac{1}{L}\sum_{q'>0}\sqrt{\frac{q'L}{2\pi}}\left(e^{-i\,q'\,x'}\, b_{q',R}\right)$$

$$\frac{1}{2}\int dx\int dx'\,\mathrm{v}(x-x')\,\rho_R(x)\rho_R(x') = \frac{1}{2}\sum_{q>0}\mathrm{v_q}\frac{q}{2\pi}b_{q,R}\,b^\dagger_{q,R} + \frac{1}{2}\sum_{q>0}\mathrm{v_q}\frac{q}{2\pi}\,b^\dagger_{q,R}\,b_{q,R}$$

We postulate that $\mathrm{v_q} = 0\ when\ |q| > \Lambda$. This allows us to write,

$$V_{RR} = \frac{1}{2}\int dx\int dx'\,\mathrm{v}(x-x')\,\rho_R(x)\rho_R(x') = const. + \sum_{q>0}\mathrm{v_q}\frac{q}{2\pi}\,b^\dagger_{q,R}\,b_{q,R}$$

Similarly,

$$V_{LL} = \frac{1}{2}\int dx\int dx'\,\mathrm{v}(x-x')\,\rho_L(x)\rho_L(x') = const. + \sum_{q>0}\mathrm{v_q}\frac{q}{2\pi}\,b^\dagger_{-q,L}\,b_{-q,L}$$

Now,

$$\frac{1}{2}\int dx\int dx'\,\mathrm{w}(x-x')\,(\rho_R(x)\rho_L(x') + \rho_L(x)\rho_R(x'))$$

This means, $\mathrm{V}_{RL} = \frac{1}{2}\int dx \int dx'\, \mathrm{w}(x-x')\,(\rho_R(x)\rho_L(x') + \rho_L(x)\rho_R(x')) = \frac{1}{2}\sum_{q>0} w_q \;\frac{q}{2\pi}\;(\,b_{q,R}\; b_{-q,L} + \; b^{\dagger}_{-q,L}\; b^{\dagger}_{q,R})$

The total Hamiltonian of both right and left movers together with mutual interactions is,

$$\mathrm{H} = :H_R: + :H_L: + V_{RR} + V_{LL} + V_{RL}$$

$$:H_R: = \sum_{q>0} \mathrm{v}_F q\; b^{\dagger}_{q,R}\; b_{q,R} + \mathrm{v}_F \frac{\pi :N:_R^2}{L}$$

$$:H_L: = \sum_{q>0} \mathrm{v}_F q\; b^{\dagger}_{-q,L}\; b_{-q,L} + \mathrm{v}_F \frac{\pi :N:_L^2}{L}$$

# Diagonalization of the bosonized Hamiltonian

The full Hamiltonian including interactions is,

$$H = \sum_{q>0} \left( \mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi} \right) q \, b^{\dagger}_{q,R} \, b_{q,R} + \mathrm{v}_F \frac{\pi :N:^2_R}{L}$$

$$+ \sum_{q>0} \left( \mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi} \right) q \, b^{\dagger}_{-q,L} \, b_{-q,L} + \mathrm{v}_F \frac{\pi :N:^2_L}{L}$$

$$+ \frac{1}{2} \sum_{q>0} w_q \, \frac{q}{2\pi} \left( b_{q,R} \, b_{-q,L} + b^{\dagger}_{-q,L} \, b^{\dagger}_{q,R} \right)$$

We postulate that the diagonalized form is as shown below,

$$H = \Omega_{q,1}\, d^{\dagger}_{q,1} d_{q,1} + \Omega_{q,2}\, d^{\dagger}_{q,2} d_{q,2} + \mathrm{v}_F \frac{\pi :N:^2_R}{L} + \mathrm{v}_F \frac{\pi :N:^2_L}{L}$$

where,

$$d_{q,1} = [d_{q,1}\,, b^{\dagger}_{q,R}] b_{q,R} + [b_{-q,L}, d_{q,1}] b^{\dagger}_{-q,L}$$

and,

$$d_{q,2} = [d_{q,2}\,, b^{\dagger}_{-q,L}] b_{-q,L} + [b_{q,R}, d_{q,2}] b^{\dagger}_{q,R}$$

$$\Omega_{q,1}\, d_{q,1} = \left[d_{q,1}\,, H\right] = \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\, [d_{q,1}\,, b^\dagger_{q,R}\,]\;\; b_{q,R}$$
$$+ \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\; b^\dagger_{-q,L}\, [d_{q,1}\,, b_{-q,L}]$$
$$+ \frac{1}{2} w_q\;\; \frac{q}{2\pi}\;\, (\, b_{q,R}\, [d_{q,1}\,, b_{-q,L}\,] + b^\dagger_{-q,L}\, [d_{q,1}\,, b^\dagger_{q,R}\,]\,)$$

and

$$\Omega_{q,2}\, d_{q,2} = \left[d_{q,2}\,, H\right] = \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\, [d_{q,2}\,, b^\dagger_{-q,L}\,]\;\; b_{-q,L}$$
$$+ \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\; b^\dagger_{q,R}\, [d_{q,2}\,, b_{q,R}]$$
$$+ \frac{1}{2} w_q\;\; \frac{q}{2\pi} (\, [d_{q,2}\,, b_{q,R}\,] b_{-q,L} + \; [d_{q,2}\,, b^\dagger_{-q,L}\,] b^\dagger_{q,R})$$

$$\Omega_{q,1}\,[d_{q,1}, b^{\dagger}_{q,R}] = \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\,[d_{q,1}\,, b^{\dagger}_{q,R}] + \frac{1}{2} w_q \;\frac{q}{2\pi}\;[d_{q,1}\,, b_{-q,L}]$$

and

$$\Omega_{q,1}[d_{q,1}\,, b_{-q,L}] = -\left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\;[d_{q,1}\,, b_{-q,L}] - \frac{1}{2} w_q \;\frac{q}{2\pi}\;[d_{q,1}\,, b^{\dagger}_{q,R}]$$

and

$$\Omega_{q,2}\,[d_{q,2}, b^{\dagger}_{-q,L}] = \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\,[d_{q,2}\,, b^{\dagger}_{-q,L}] + \frac{1}{2} w_q \;\frac{q}{2\pi}[d_{q,2}\,, b_{q,R}]$$

and

$$\Omega_{q,2}[d_{q,2}\,, b_{q,R}] = -\left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right) q\;[d_{q,2}\,, b_{q,R}] - \frac{1}{2} w_q \;\frac{q}{2\pi}[d_{q,2}\,, b^{\dagger}_{-q,L}]$$

This means, $\Omega_{q,\nu} = \left(q^2 \left(\mathrm{v}_F + \frac{\mathrm{v}_q}{2\pi}\right)^2 - \left(\frac{1}{2} w_q \; \frac{q}{2\pi}\right)^2\right)^{\frac{1}{2}}$

$$(\Omega_q + \mathrm{v}_F q + q\frac{\mathrm{v}_q}{2\pi})[d_{q,1}, b_{-q,L}] = -\frac{1}{2} w_q \; \frac{q}{2\pi} \; [d_{q,1}, b^{\dagger}_{q,R}] = \mathrm{c}_{q,1}$$

$$(\Omega_q + \mathrm{v}_F q + q\frac{\mathrm{v}_q}{2\pi}) [d_{q,2}, b_{q,R}] = -\frac{1}{2} w_q \; \frac{q}{2\pi} [d_{q,2}, b^{\dagger}_{-q,L}] = \mathrm{c}_{q,2}$$

$$[d_{q,1}, b_{-q,L}] = \frac{c_{q,1}}{(\Omega_q + v_F q + q\frac{v_q}{2\pi})}$$

$$[d_{q,2}, b_{q,R}] = \frac{c_{q,2}}{(\Omega_q + v_F q + q\frac{v_q}{2\pi})}$$

$$[d_{q,1}, b^{\dagger}_{q,R}] = -\frac{c_{q,1}}{\frac{1}{2} w_q \quad \frac{q}{2\pi}}$$

$$[d_{q,2}, b^{\dagger}_{-q,L}] = -\frac{c_{q,2}}{\frac{1}{2} w_q \quad \frac{q}{2\pi}}$$

$$d_{q,1} = -\frac{c_{q,1}}{\frac{1}{2} w_q \quad \frac{q}{2\pi}} b_{q,R} - \frac{c_{q,1}}{(\Omega_q + v_F q + q \frac{v_q}{2\pi})} b^{\dagger}_{-q,L}$$

and,

$$d_{q,2} = -\frac{c_{q,2}}{\frac{1}{2} w_q \quad \frac{q}{2\pi}} b_{-q,L} - \frac{c_{q,2}}{(\Omega_q + v_F q + q \frac{v_q}{2\pi})} b^{\dagger}_{q,R}$$

But $[d_{q,1}, d^{\dagger}_{q,1}] = [d_{q,2}, d^{\dagger}_{q,2}] = 1 , [d_{q,1}, d_{q,2}] = 0$

$$c_{q,1} = \left( \frac{1}{\left( \frac{1}{\frac{1}{2} w_q \quad \frac{q}{2\pi}} \right)^2 - \left( \frac{1}{(\Omega_q + v_F q + q \frac{v_q}{2\pi})} \right)^2} \right)^{\frac{1}{2}}$$

# Correlation Functions of Interacting Systems

$$\rho_R(x) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\, e^{-i\,q\,x}\, b_{q,R} + \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\, e^{i\,q\,x}\, b^{\dagger}_{q,R}$$

$$\rho_L(x) = \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\, e^{i\,q\,x}\, b_{-q,L} + \frac{1}{L}\sum_{q>0}\sqrt{\frac{qL}{2\pi}}\, e^{-i\,q\,x}\, b^{\dagger}_{-q,L}$$

Of interest are correlation functions such as $< \rho_R\,(x,t)\,\rho_R(x',t') >$ and $< \rho_R\,(x,t)\,\rho_L(x',t') >$ etc. From the above relations it is clear that evaluating this is possible if we are first able to evaluate correlations such as $< b_{q,R}\,(t) b^{\dagger}_{q,R}(t') >$, $< b_{q,R}\,(t)\, b_{-q,L}(t') >$

$$b_{q,R} = [b_{q,R}, d^{\dagger}_{q,1}]d_{q,1} - [b_{q,R}, d_{q,2}]d^{\dagger}_{q,2}$$

and,

$$b_{-q,L} = [b_{-q,L}, d^{\dagger}_{q,2}]d_{q,2} - [b_{-q,L}, d_{q,1}]d^{\dagger}_{q,1}$$

So that

$$< b_{q,R}\ (t)b^{\dagger}_{q,R}(t') >= [b_{q,R}, d^{\dagger}_{q,1}]^2 < d_{q,1}(t)\ d^{\dagger}_{q,1}(t') >$$

$$= \ [b_{q,R}, d^{\dagger}_{q,1}]^2\ \mathrm{e}^{-i\,\Omega_q\,(t-t')}$$

and

$$< b_{q,R}\ (t)b_{-q,L}(t') > = [b_{q,R}, d^{\dagger}_{q,1}][b_{-q,L}, d^{\dagger}_{q,2}]\mathrm{e}^{-i\,\Omega_q\,(t-t')}$$

# The central claim of bosonization

One of the central claims of bosonization of chiral fermions is that the two operators below are identical in all respects.

$$:P_R:=\sum_{q>0} q\; b^{\dagger}_{q>0,R}\; b_{q>0,R}+\frac{\pi :N:_R^2}{L}$$

The claim is that this operator is identical to

$$:P_R: =\sum_{p>0} p\,(\,c^{\dagger}_{p,R}\; c_{p,R}\; +\; c_{-p,R}\; c^{\dagger}_{-p,R})$$

where

$$:N_R: =\sum_{p>0} (\,c^{\dagger}_{p,R}\; c_{p,R}\; -c_{-p,R}\; c^{\dagger}_{-p,R})$$

$$b_{q>0,R} = \sqrt{\frac{2\pi}{qL}}\sum_{k} c^{\dagger}_{k,R}\; c_{k+q,R}$$

If $N_R$ is the eigenvalue of $:N_R:$ , the Virasoro primary state $|\mathrm{G}>$ has these properties ( $k_F = \frac{2\pi N_R}{L}$ )

$$c^{\dagger}_{p<k_F,R}\,|G> = 0 \;; \quad c_{p>k_F,R}|G> = 0$$

$$b_{q>0,R}|\mathrm{G}> = 0$$

$$:N_R:|G> = N_R\,|G>$$

Examine,

$$\left[c_{p>0,R}\ ,: P_R :\right] = [c_{p,R}, \sum_{q>0} q\ b^{\dagger}_{q>0,R}\ b_{q>0,R} + \frac{\pi : N :^{2}_{R}}{L}]$$

The claim is that this is the same as

$$\left[c_{p>0,R}\ ,: P_R :\right] = p\ c_{p>0,R}$$

$$[c_{p>0,R}\,,:P_R:] = p\,c_{p,R} = \sum_{q>0} q\sqrt{\frac{2\pi}{qL}}\,c_{p-q,R}\;b_{q>0,R} + \sum_{q>0} q\,b^{\dagger}_{q>0,R}\sqrt{\frac{2\pi}{qL}}\,c_{p+q,R} + \frac{\pi\,c_{p,R}:N:_R}{L} + \frac{\pi:N:_R\;c_{p,R}}{L}$$

Pre-multiply by a creation operator

$$p c^{\dagger}_{p,R} c_{p,R} = \sum_{q>0} q\sqrt{\frac{2\pi}{qL}}\,c^{\dagger}_{p,R} c_{p-q,R}\;b_{q>0,R} + \sum_{q>0} q\,c^{\dagger}_{p,R}\,b^{\dagger}_{q>0,R}\sqrt{\frac{2\pi}{qL}}\,c_{p+q,R} + \frac{\pi c^{\dagger}_{p,R} c_{p,R}:N:_R}{L} + \frac{\pi c^{\dagger}_{p,R}:N:_R\;c_{p,R}}{L}$$

Post-multiply by the same creation operator

$$p\,c_{p,R} c^{\dagger}_{p,R} = \sum_{q>0} q\sqrt{\frac{2\pi}{qL}}\,c_{p-q,R}\;b_{q>0,R} c^{\dagger}_{p,R} + \sum_{q>0} q\,b^{\dagger}_{q>0,R}\sqrt{\frac{2\pi}{qL}}\,c_{p+q,R}\,c^{\dagger}_{p,R} + \frac{\pi\,c_{p,R}:N:_R\;c^{\dagger}_{p,R}}{L} + \frac{\pi:N:_R\;c_{p,R} c^{\dagger}_{p,R}}{L}$$

Adding the two

$$p = \sum_{q>0} q\sqrt{\frac{2\pi}{qL}}\,c_{p-q,R}\;[b_{q>0,R}, c^{\dagger}_{p,R}] + \sum_{q>0} q[c^{\dagger}_{p,R}\,, b^{\dagger}_{q>0,R}]\sqrt{\frac{2\pi}{qL}}\,c_{p+q,R} + \frac{\pi\,c_{p,R}\,[:N:_R\,, c^{\dagger}_{p,R}]}{L} + \frac{\pi[c^{\dagger}_{p,R}\,,:N:_R\,]\,c_{p,R}}{L} + \frac{2\pi\;:N:_R}{L}$$

$$p = \sum_{q>0} q\sqrt{\frac{2\pi}{qL}}\, c_{p-q,R}\ [b_{q>0,R}, c^{\dagger}_{p,R}] + \sum_{q>0} q[c^{\dagger}_{p,R}, b^{\dagger}_{q>0,R}]\sqrt{\frac{2\pi}{qL}}\, c_{p+q,R} + \frac{\pi\, c_{p,R}\,[:N:_R, c^{\dagger}_{p,R}]}{L} + \frac{\pi[c^{\dagger}_{p,R}, :N:_R\,]\, c_{p,R}}{L} + \frac{2\pi\ :N:_R}{L}$$

$$p = \sum_{q>0} \frac{2\pi}{L}\, c_{p-q,R}\ c^{\dagger}_{p-q,R} - \sum_{q>0} \frac{2\pi}{L}\, c^{\dagger}_{p+q,R}\ c_{p+q,R} + \frac{\pi\, c_{p,R}\,[:N:_R, c^{\dagger}_{p,R}]}{L} + \frac{\pi[c^{\dagger}_{p,R}, :N:_R\,]\, c_{p,R}}{L} + \frac{2\pi\ :N:_R}{L}$$

$$:N_R: = \sum_{p>0} (\, c^{\dagger}_{p,R}\, c_{p,R} - c_{-p,R}\, c^{\dagger}_{-p,R})$$

$$p = \sum_{q>0} \frac{2\pi}{L}\, c_{p-q,R}\ c^{\dagger}_{p-q,R} - \sum_{q>0} \frac{2\pi}{L}\, c^{\dagger}_{p+q,R}\ c_{p+q,R} + \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi\ :N:_R}{L}$$

$$p = \sum_{q>0} \frac{2\pi}{L}\, c_{p-q,R}\, c^{\dagger}_{p-q,R} - \sum_{q>0} \frac{2\pi}{L} c^{\dagger}_{p+q,R}\, c_{p+q,R} + \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi}{L} \sum_{p'>0} ( c^{\dagger}_{p',R}\, c_{p',R} - c_{-p',R}\, c^{\dagger}_{-p',R})$$

$$p = \sum_{p'>-p} \frac{2\pi}{L}\, c_{-p',R}\, c^{\dagger}_{-p',R} - \sum_{p'>p} \frac{2\pi}{L} c^{\dagger}_{p',R}\, c_{p',R} + \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi}{L} \sum_{p'>0} c^{\dagger}_{p',R}\, c_{p',R} - \frac{2\pi}{L} \sum_{p'>0} c_{-p',R}\, c^{\dagger}_{-p',R}$$

$$p = \sum_{0>p'>-p} \frac{2\pi}{L}\, c_{-p',R}\, c^{\dagger}_{-p',R} + \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi}{L} \sum_{p\geq p'>0} c^{\dagger}_{p',R}\, c_{p',R}$$

$$p = \sum_{0<p'<p} \frac{2\pi}{L}\, c_{p',R}\, c^{\dagger}_{p',R} + \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi}{L} \sum_{p\geq p'>0} c^{\dagger}_{p',R}\, c_{p',R}$$

$$p = \sum_{0<p'\leq p} \frac{2\pi}{L}\, c_{p',R}\, c^{\dagger}_{p',R} - \frac{\pi\, c_{p,R} c^{\dagger}_{p,R}}{L} - \frac{\pi\, c^{\dagger}_{p,R}\, c_{p,R}}{L} + \frac{2\pi}{L} \sum_{p\geq p'>0} c^{\dagger}_{p',R}\, c_{p',R}$$

$$p \;=\sum_{0< \;p'\leq p}\frac{2\pi}{L}\,c_{p',R}\;c^{\dagger}_{p',R}-\frac{\pi}{L}+\frac{2\pi}{L}\sum_{p\geq p'>0}c^{\dagger}_{p',R}\,c_{p',R}\;=\sum_{0< \;p'\leq p}\frac{2\pi}{L}\;-\frac{\pi}{L}$$

Write $p=\frac{(2n_p+1)\pi}{L}$ ; $p'=\frac{(2\,n'+1)\pi}{L}$. The above sum is over integers from $n'= \;0,1,2,,\ldots\ldots,n_p$

$$p \;=\frac{2\pi}{L}(n_p+1)-\frac{\pi}{L}=p$$

**THUS THE PROOF IS NOW COMPLETE. IT IS NOT KNOWN TO MOST PEOPLE WORKING IN THE FIELD. MOST IN FACT DON'T EVEN REALISE THE IMPORTANCE OF PROVING THIS RESULT.**

SEE PDF ATTACHMENTS OF PROF. DUNCAN HALDANE’S PROOFS

# Finding the Full Green Function

$$\Psi_R^\dagger(x,t) = \frac{1}{\sqrt{L}}\, e^{i\,\varphi_R^+(x,t)}\; e^{i\,\theta_R} e^{i\,\varphi_R^-(x,t)}$$

$$\varphi_R^-(x,t)\; = \pi\, N_R \frac{x}{L} + i \sum_{q>0} \frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x}\;\; b_{q,R}(t)$$

$$\varphi_R^+(x,t)\; = \pi\, N_R \frac{x}{L} - i \sum_{q>0} \frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x}\;\; b_{q,R}^\dagger(t)$$

$$\Psi_R\;(x,t) = \frac{1}{\sqrt{L}}\, e^{-i\,\varphi_R^+(x,t)}\; e^{-i\,\theta_R} e^{-i\,\varphi_R^-(x,t)}$$

$$e^{i\,\theta_R}\;N_R\,|N_R> = \;N_R\,|1 + N_R>$$

$$N_R e^{i\,\theta_R}\;|N_R> = (N_R + 1)|1 + N_R>$$

$$e^{i\,\theta_R}\;|N_R> = \;|1 + N_R>$$

$$N_R e^{i\,\theta_R}\; - e^{i\,\theta_R}\;N_R = e^{i\,\theta_R}$$

$$e^{-i\,\theta_R}\;N_R\;e^{i\,\theta_R} = \;N_R + 1$$

$$< \Psi_R^\dagger (x',t')\Psi_R (x,t) > = \frac{1}{L} < e^{i\,\varphi_R^+(x',t')}\; e^{i\,\theta_R} e^{i\,\varphi_R^-(x',t')}\; e^{-i\,\varphi_R^+(x,t)} e^{-i\,\theta_R} e^{-i\,\varphi_R^-(x,t)} >$$

$$< \Psi_R^\dagger (x',t')\Psi_R (x,t) > = \frac{1}{L} < e^{i\,\varphi_R^+(x',t')}\; e^{i\, e^{i\,\theta_R}\,\varphi_R^-(x',t') e^{-i\,\theta_R}}\; e^{-i\, e^{i\,\theta_R}\,\varphi_R^+(x,t) e^{-i\,\theta_R}} e^{-i\,\varphi_R^-(x,t)} >$$

Since

$$e^{i\,\theta_R}\, f(N_R)\; e^{-i\,\theta_R} = f\left(e^{i\,\theta_R}\; N_R\, e^{-i\,\theta_R}\right) = f(N_R - 1)$$

$$< \Psi_R^\dagger (x',t')\Psi_R (x,t) >$$

$$= \frac{1}{L} e^{-i\pi \frac{(x'-x)}{L}} < e^{i\varphi_R^+(x',t')} e^{i\varphi_R^-(x',t')} e^{-i\varphi_R^+(x,t)} e^{-i\varphi_R^-(x,t)} >$$

$$< \Psi_R^\dagger (x',t')\Psi_R (x,t) >$$

$$= \frac{1}{L} e^{i\pi (2N_R-1)\frac{(x'-x)}{L}} < e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}} e^{iqx'} b_{q,R}^\dagger(t')} e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}} e^{-iqx'} b_{q,R}(t')}$$

$$e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}} e^{iqx} b_{q,R}^\dagger(t)} e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}} e^{-iqx} b_{q,R}(t)} >$$

$$< \Psi_R^\dagger\,(x',t')\Psi_R\,(x,t) >$$

$$= \frac{1}{L}\, e^{i\,\pi\,(2N_R-1)\,\frac{(x'-x)}{L}} < e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x'}\;\; b_{q,R}^\dagger(t')}\;\; e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x'}\;\; b_{q,R}(t')}$$

$$e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x}\;\; b_{q,R}^\dagger(t)}\;\; e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x}\;\; b_{q,R}(t)} >$$


$$< \Psi_R^\dagger\,(x',t')\Psi_R\,(x,t) >$$

$$= \frac{1}{L}\, e^{i\,\pi\,(2N_R-1)\,\frac{(x'-x)}{L}} < e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x'}\;\; b_{q,R}^\dagger(t')}\;\; e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x'}\;\; b_{q,R}(t')}$$

$$e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x}\;\; b_{q,R}^\dagger(t)}\;\; e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x}\;\; b_{q,R}(t)} >$$

$$< \Psi_R^\dagger (x',t')\Psi_R (x,t) >$$

$$= \frac{1}{L}\, e^{i\,\pi\,(2N_R-1)\,\frac{(x'-x)}{L}} < e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x'}\;b_{q,R}^\dagger(t')}\; e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x'}\;b_{q,R}(t')}$$

$$e^{-\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{i\,q\,x}\;b_{q,R}^\dagger(t)}\; e^{\sum_{q>0}\frac{\sqrt{2\pi}}{\sqrt{qL}}\, e^{-i\,q\,x}\;b_{q,R}(t)} >$$

$$b_{q,R}(t) = \left[d_{q,2}\,,b_{-q,L}^\dagger\right]d_{q,1}e^{-i\,\Omega_q\,t} - [b_{-q,L},d_{q,1}]d_{q,2}^\dagger\,e^{i\,\Omega_q\,t}$$

$$b_{q,R}(t) = \left[d_{q,2}\,, b^{\dagger}_{-q,L}\right] d_{q,1} e^{-i\,\Omega_q\,t} - [b_{-q,L}, d_{q,1}] d^{\dagger}_{q,2}\, e^{i\,\Omega_q\,t}$$

$$b^{\dagger}_{q,R}(t') = \left[d_{q,2}\,, b^{\dagger}_{-q,L}\right] d^{\dagger}_{q,1} e^{i\,\Omega_q\,t'} - [b_{-q,L}, d_{q,1}] d_{q,2}\, e^{-i\,\Omega_q\,t'}$$

$$< b_{q,R}(t) b_{q',R}(t') > = 0$$

$$< b_{q,R}(t) b^{\dagger}_{q',R}(t') > = \delta_{q,q'} \left[d_{q,2}\,, b^{\dagger}_{-q,L}\right]\left[d_{q,2}\,, b^{\dagger}_{-q,L}\right] e^{-i\,\Omega_q\,(t-t')}$$

$$< b^{\dagger}_{q',R}(t') b_{q,R}(t) > = \delta_{q,q'} [b_{-q,L}, d_{q,1}][b_{-q,L}, d_{q,1}] e^{i\,\Omega_q\,(t-t')}$$

$$< \Psi_R^\dagger(x',t')\Psi_R(x,t) >$$

$$= \frac{1}{L}\, e^{i\,\pi\,(2N_R-1)\,\frac{(x'-x)}{L}}\, e^{-\sum_{q>0}\frac{2\pi}{q\,L}<b_{q,R}^\dagger(t')b_{q,R}(t')>}$$

$$e^{\sum_{q>0}\frac{2\pi}{q\,L}e^{i\,q\,(x'-x)}\,<b_{q,R}^\dagger(t')b_{q,R}(t)>}$$

$$e^{\sum_{q>0}\frac{2\pi}{q\,L}\,e^{i\,q\,(x-x')}<b_{q,R}(t')b_{q,R}^\dagger(t)>}$$

$$e^{-\sum_{q>0}\frac{2\pi}{q\,L}<b_{q,R}^\dagger(t)b_{q,R}(t)>}$$

$< b_{q,R}^\dagger(t')b_{q,R}(t') >$ *etc. have all been caculated previously*

# CONCLUSIONS

- Chiral bosonization is a powerful tool for studying homogeneous interacting fermions when the total number of right and left movers are separately conserved.

- Although the formalism relies on exact mathematical identities, studying systems which do not separately conserve the number of right movers or left movers but only the total number of right and left movers is difficult. This latter system occurs frequently for example when we consider “backward scattering” where there is a sign change in the momentum of fermions after scattering.

- Therefore although the formalism when such processes are present may be formally written down, using them is unwieldy and relies on approximations.

- This is the reason why in our group we invented **“NON-CHIRAL BOSONIZATION”**


BUT THAT IS ANOTHER STORY .....